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A Boundary Element Analysis of Electromagnetic Phenomena Using Twisted Coordinates

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ABSTRACT

This paper describes a boundary element analysis of electromagnetic fields with helical symmetry. A two dimensional integral equation for the scalar Helmholtz-type equation is derived using a spatial symmetry in the general curvilinear coordinates. An integral equation with helical symmetry is obtained from the above integral equation using twisted coordinates. The boundary element method is applied to the analyses of potential fields in a helical column, electromagnetic fields in a twisted waveguide and magnetohydrodynamic equilibria in a helical vessel.

INTRODUCTION

In computational analysis of field problems, spatial symmetries such as translational and axial symmetry often allow us to make considerable reduction of computer memories and computing times. For this reason, a number of numerical schemes have been studied especially for fields with translational and axial symmetries. On the other hand, we also find helical symmetry in many engineering devices such as stellarator-type nuclear fusion machines [1], twisted waveguides [2], optical fibers [3] and helical antennas [4]. For the analysis of the electromagnetic fields in those devices, it is convenient to employ the twisted coordinates (X, Y, Z) in which the axes X and Y rotate with the rotation of cross section [2]. Figure 1 illustrates a straight helical system of a helical pitch h and the twisted coordinates (X, Y, Z), in which the fields can be described in two dimension when one can assume some dependence of physical quantities on the axis Z, usually $exp(-j\beta Z)$. (When $\beta=0$, the system is called helically symmetric system.)

In this paper, we submit a boundary element method for electromagnetic phenomena with helical symmetry. The main purpose of this paper is to show the validity and usefulness of the boundary element analysis of helical electromagnetic field problems.

The remainder of this paper is organized as follows. We first derive a two dimensional boundary integral equation (BIE) from a Helmholtz-type differential equation in the general curvilinear coordinates. Next, from the above equation, we introduce a BIE described in the twisted coordinates for the analysis of electromagnetic problems with helical symmetry. Moreover, the present method

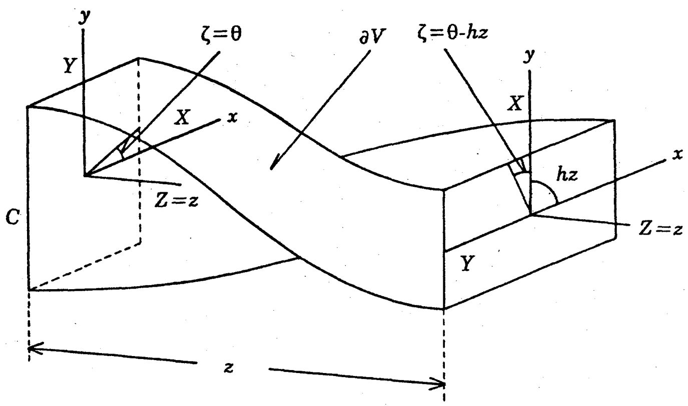

Fig. 1 Straight helical system

is applied to the analyses of potential fields in a helical column, electromagnetic fields in a twisted waveguide and magnetohydrodynamic (MHD) equilibria in a helical vessel.

FORMULATION

Two dimensional BIE in the general curvilinear coordinates

Let us consider the Helmholtz-type equation

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\left(g^{ij}\sqrt{g}K\frac{\partial\Psi}{\partial u^j}\right)+k^2\Psi$$
$$\equiv L\Psi+k^2\Psi=F\,, \tag{1}$$

in the coordinates (u^1, u^2, u^3), where g^{ij} denotes the contravariant metric tensor, $g=1/det\, g^{ij}$, $K=K(u^1, u^2)$ and $F=f(u^1, u^2, \Psi)$. Note that, in eq. (1), the operator L corresponds to the Laplacian ∇^2 when the function K is unity. We here assume that $\Psi=\psi(u^1, u^2)exp(-j\beta u^3)$ and $F=f(u^1, u^2, \psi)exp(-j\beta u^3)$.

Multiplying eq. (1) by the fundamental solution $\Psi^*=\psi^*(u^1, u^2)exp(j\beta u^3)$ which satisfies

$$\frac{1}{\sqrt{g}}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial u^i}\left(g^{ij}\sqrt{g}K\frac{\partial\psi^*}{\partial u^j}\right)+\frac{4\pi}{\sqrt{g}}\delta(u^1-u^1_0)\,\delta(u^2-u^2_0)$$
$$\equiv L_t\psi^*+\frac{4\pi}{\sqrt{g}}\delta(u^1-u^1_0)\,\delta(u^2-u^2_0)=0\,, \tag{2}$$

and integrating over a region V, enclosed by a boundary ∂V, with a period of $2\pi/\beta$ in u^3 direction, yields

$$C_0\psi_0\int du^3=\int_{\partial V}K\Psi^*\frac{\partial\Psi}{\partial n}dS-\int_{\partial V}K\Psi\frac{\partial\Psi^*}{\partial n}dS+\int_V[\Psi(L-L_t+k^2)-F]\Psi^*dV, \tag{3}$$

where $\partial/\partial n = \mathbf{n}\cdot\nabla$, $\mathbf{n}$ denotes the unit vector in the normal direction of the surface ∂V and C_0 is the geometric factor related to the Cauchy principal value of the boundary integrals and if the boundary ∂V is smooth, then $C_0 = 2\pi$. Moreover, the surface element dS and operator $\partial/\partial n$ in eq. (3) can be expressed as

$$dS = |\mathbf{N}|\, ds\, du^3, \tag{4}$$

$$\frac{\partial}{\partial n} = \frac{\mathbf{N}}{|\mathbf{N}|}\cdot \mathbf{e}^i \frac{\partial}{\partial u^i}, \tag{5}$$

where

$$\mathbf{N} \equiv \left(\frac{du^1}{ds}\mathbf{e}_1 + \frac{du^2}{ds}\mathbf{e}_2\right) \times \mathbf{e}_3, \tag{6}$$

and ds denotes the differential line element on the intersection C between surface of u^3=constant and ∂V, and $\mathbf{e}_i$ and $\mathbf{e}^i$ are the unitary and reciprocal unitary vectors, respectively.

From eqs. (3) - (6), we can derive the two dimensional BIE as follows :

$$C_0\psi_0 = \oint_C K\psi^*\, q\, P\, ds - \oint_C K\psi\, Q(\psi^*)\, ds + \int_\Omega [\psi R(\psi^*) - f\psi^*\sqrt{g}]\, du^1 du^2, \tag{7}$$

where

$$P \equiv \sqrt{g\left[g^{22}\left(\frac{du^1}{ds}\right)^2 - 2g^{12}\frac{du^1}{ds}\frac{du^2}{ds} + g^{11}\left(\frac{du^2}{ds}\right)^2\right]}, \tag{8}$$

$$Q \equiv \sqrt{g}\left[-\frac{du^1}{ds}\left(g^{12}\frac{\partial}{\partial u^1} + g^{22}\frac{\partial}{\partial u^2} + j\beta g^{23}\right) + \frac{du^2}{ds}\left(g^{11}\frac{\partial}{\partial u^1} + g^{12}\frac{\partial}{\partial u^2} + j\beta g^{13}\right)\right], \tag{9}$$

$$R \equiv j\beta\left[\sqrt{g}K\left(g^{13}\frac{\partial}{\partial u^1} + g^{23}\frac{\partial}{\partial u^2}\right) + \left(\frac{\partial}{\partial u^1}g^{13} + \frac{\partial}{\partial u^2}g^{23}\right)\sqrt{g}K\right] + \sqrt{g}(k^2 - \beta^2 g^{33}K), \tag{10}$$

and $q \equiv \partial\Psi/\partial n$. When we specify the system, that is, determine the metric tensor g^{ij}, we can readily obtain the two dimensional BIE for the system from eq. (7).

<u>BIE with helical symmetry</u>

For the analysis of straight helical systems, we introduce the twisted coordinates (X, Y, Z) [2]

$$\begin{aligned} X &= x\cos(hz) + y\sin(hz), \\ Y &= y\cos(hz) - x\sin(hz), \\ Z &= z, \end{aligned} \tag{11}$$

where (x, y, z) are the Cartesian coordinates. Figure 2 illustrates the relation between the twisted and Cartesian coordinates. In a straight helical system, as

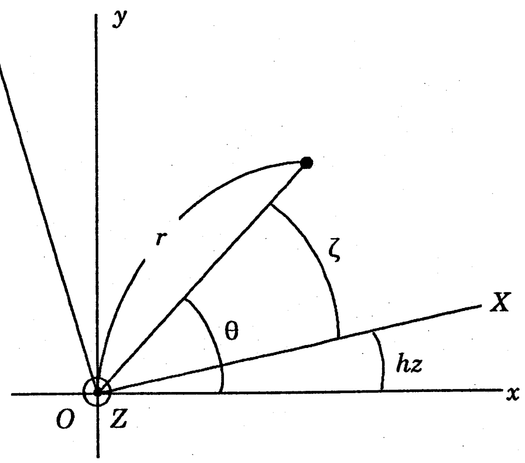

Fig. 2. Twisted Coordinates (X, Y, Z)

shown in Fig. 1, physical quantities may be dependent on Z in the form $exp(-j\beta Z)$. Hence we can apply the formulation in the previous section to these systems.

The metric tensor for the twisted coordinates g^{ij} is given by

$$\left[g^{ij}\right] = \begin{bmatrix} 1+h^2Y^2 & -h^2XY & hY \\ -h^2XY & 1+h^2X^2 & -hX \\ hY & -hX & 1 \end{bmatrix}, \tag{12}$$

and g is proved to be unity.

Substituting eq. (12) into eq. (1), it can be shown that the Helmholtz-type differential equation of our interest takes the following form in the twisted coordinates

$$\begin{aligned} &(1+h^2Y^2)K\frac{\partial^2\psi}{\partial X^2} - 2h^2XYK\frac{\partial^2\psi}{\partial X\partial Y} + (1+h^2X^2)K\frac{\partial^2\psi}{\partial Y^2} \\ &-\left[K(2j\beta hY + h^2X) - (1+h^2Y^2 - j\beta hY)\frac{\partial K}{\partial X} + h^2XY\frac{\partial K}{\partial Y}\right]\frac{\partial\psi}{\partial X} \\ &+\left[K(2j\beta hX - h^2Y) + (1+h^2X^2 + j\beta hX)\frac{\partial K}{\partial Y} - h^2XY\frac{\partial K}{\partial X}\right]\frac{\partial\psi}{\partial Y} + (k^2 - K\beta^2)\psi = f. \end{aligned} \tag{13}$$

Moreover, the boundary integral equation to be numerically solved can be derived from eqs. (7)-(10) and (12) as follows :

$$C_0\psi_0 = \oint_C K\psi^* q\,[\,1 + h^2(XX' + YY')^2\,]^{1/2}\, ds$$
$$- \oint_C K\psi\left[X'\left\{h^2XY\frac{\partial}{\partial X} - (1+h^2X^2)\frac{\partial}{\partial Y} + j\beta hX\right\}\right.$$
$$\left.+Y'\left\{-h^2XY\frac{\partial}{\partial Y} + (1+h^2Y^2)\frac{\partial}{\partial X} + j\beta hY\right\}\right]\psi^*\, ds$$
$$+\int_\Omega \left[j\beta h\psi\left\{2K\left(Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}\right) + Y\frac{\partial K}{\partial X} - X\frac{\partial K}{\partial Y}\right\} + \psi(k^2-\beta^2K) - f\right]\psi^*\, dXdY. \tag{14}$$

NUMERICAL RESULTS

Helically symmetric potential problem ($K=1,\ k=\beta=f=0$)

We solve a helically symmetric potential problem in a helical column with a rectangular cross-section shown in Fig.3. In this case, since the equation to be solved is reduced to the Laplace equation, the functions K, f and constant k in eq. (14) are taken to be 1.0, 0.0 (constant) and 0.0, respectively. Moreover, the constant β is set to 0.0 because of the helical symmetry.

The fundamental solution ψ^* for this case is given by

$$\psi^* = -2\log r_> + 4\sum_{m=1}^{\infty} I_m(mhr_<)K_m(mhr_>)\cos[\,m(\zeta-\zeta_0)\,], \tag{15}$$

where I_m and K_m are the modified Bessel functions of the 1st and 2nd kind and $r_> \equiv max(r, r_0)$ and $r_< \equiv min(r, r_0)$. It can be shown that the representation (15) is equivalent to the integral form [1, 6, 7]

$$\psi^* = \int_{-\infty}^{\infty}\left[\frac{1}{R} - \frac{1}{\{a^2+(z-z_0)^2\}^{1/2}}\right]dz_0, \tag{16}$$

where the integration is performed along the source on the helix $r=r_0$, $\zeta=\zeta_0$ and the distance R between the source and a field point (r, θ, z) is defined by $R\equiv\{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+(z-z_0)^2\}^{1/2}$. Here, note that θ_0 is the function of z_0, *i.e.*, $\theta_0=\zeta_0+hz_0$. In addition, the second term of the integrand in eq. (16) is the additional term which makes the integration converge and a is an arbitrary constant. A helical source and field point are illustrated in Fig. 4. Although we can clearly understand the physical meaning of the representation (16), the numerical evaluation of the integration is considerably expensive. Namely, we have to subdivide the integral domain into very small elements especially when field points are near the source because the function $1/R$ vibrates with a large amplitude in such situations. For this reason, we calculate the value of ψ^* by representation (15) in this paper. An effective scheme for evaluation of the infinite series in eq. (15) is discussed in detail in the references [8, 9].

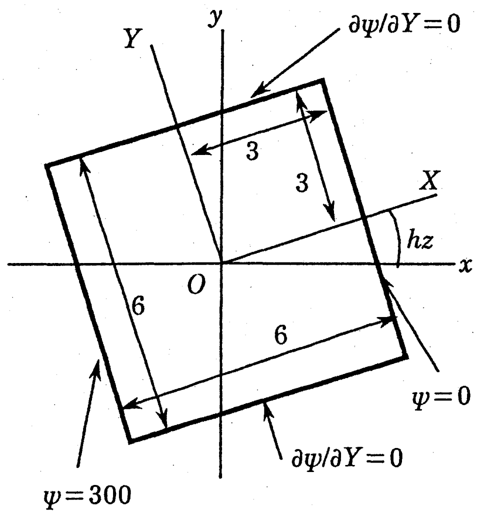

Fig. 3. Potential problem

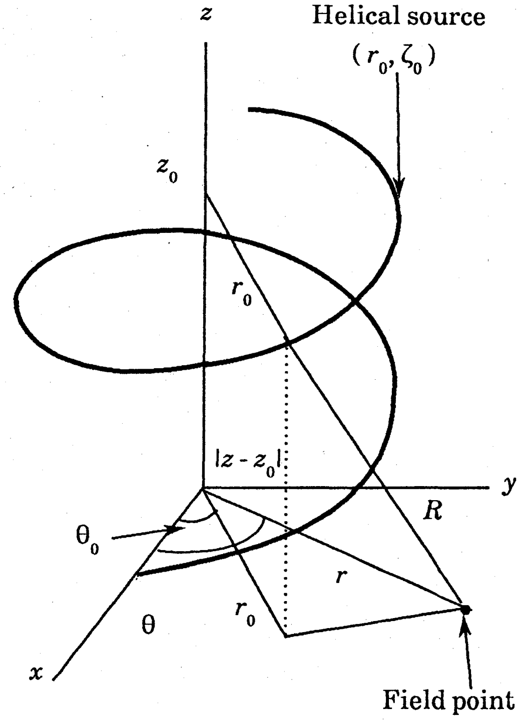

Fig. 4. Helical source

To numerically solve the integral equation (14), we subdivide the contour in Fig. 3 into 80 linear portions and use linear and constant interpolations for the potential ψ and flux q, respectively.

The solutions by boundary element method for $h=0.3$, 0.5 are summarized in Table 1, which also shows the 2nd-order perturbed solutions and finite element solutions [9]. From these results, it can be seen that the boundary element solution is in good agreement with the 2nd-order perturbed solution and the finite element solution for $h=0.3$. However, there are significant discrepancies between the numerical solutions and the 2nd-order perturbed solution for $h=0.5$. This may be caused by an insufficient convergence of the perturbation.

The present method can also be applied to open boundary potential problems with helical symmetry [10].

Electromagnetic fields in a twisted waveguide ($K=1, f=0$)

We next apply the present method to the analysis of electromagnetic fields in a twisted waveguide with a rectangular cross section shown in Fig. 5. The governing equation of the electromagnetic field in this waveguide is expressed by the scalar Helmholtz equation in an approximation [2]. Hence, in this case, the functions K and f are taken to be 1.0 and 0.0, and the constants k and β correspond to the wave number and phase constant, respectively. Moreover, the function ψ corresponds to the amplitude of the magnetic Hertzian vector $\psi \mathbf{e}^Z$ which satisfies

Table 1. Solutions to the potential problem (values of ψ for $Y=0$)

Helical pitch h	X	Zero-pitch solution	2nd-order perturbed solution	FE solution	BE solution
0.3	-1.5	225.00	218.21	218.11	218.39
0.3	0.0	150.00	150.00	150.00	150.00
0.3	1.5	75.00	81.79	81.89	81.61
0.5	-1.5	225.00	210.11	208.40	208.63
0.5	0.0	150.00	150.00	150.00	150.00
0.5	1.5	75.00	89.89	91.60	91.37

$$E^X = \frac{\partial \psi}{\partial Y}, \quad E^Y = -\frac{\partial \psi}{\partial X}, \quad E^Z = 0, \tag{17}$$

$$\mathbf{H} = \frac{j}{\omega\mu}\left[k^2 \psi \mathbf{e}^Z + \nabla\left(\frac{\partial \psi}{\partial z}\right)\right], \tag{18}$$

where ω is the angular frequency, μ the permeability, E^X, E^Y and E^Z the contravariant components of the electric field, H the magnetic field, and $\mathbf{e}^Z$ the third covariant basis vector of the twisted coordinates. Moreover, approximate boundary conditions for the rectangular guide of width a and height b are

$$\begin{aligned} E^X &= \frac{\partial \psi}{\partial Y} = 0, \qquad \text{at } Y = \pm b/2, \\ E^Y &= -\frac{\partial \psi}{\partial X} = 0, \qquad \text{at } X = \pm a/2. \end{aligned} \tag{19}$$

The dispersion relation for this waveguide can be obtained by solving the proper equation (13) for given phase constants. In this paper, the phase constant β is set to zero for simplicity. (Hence, the eigenvalue k_i corresponds to the cut-off wave number for i-th mode).

On the other hand, we have two ways to solve the scalar Helmholtz equation by BEM; the method with the fundamental solution to (a) the scalar Helmholtz equation, and (b) the Laplace equation. If we use method (a), we have to calculate the eigenvalues by a determinant search method and, in general, it is expensive to carry out. In method (b), which is sometimes called the hybrid BEM [11], the term $k^2\psi$ is treated as an inhomogeneous term and the unknown values ψ_i are at both domain and boundary. The method (b) allows us to effectively find eigenvalues because BIE (14) is reduced to a linear proper equation. For this reason, we here employ the method (b).

The resultant eigenvalues k for the lowest mode are summarized in table 2. The

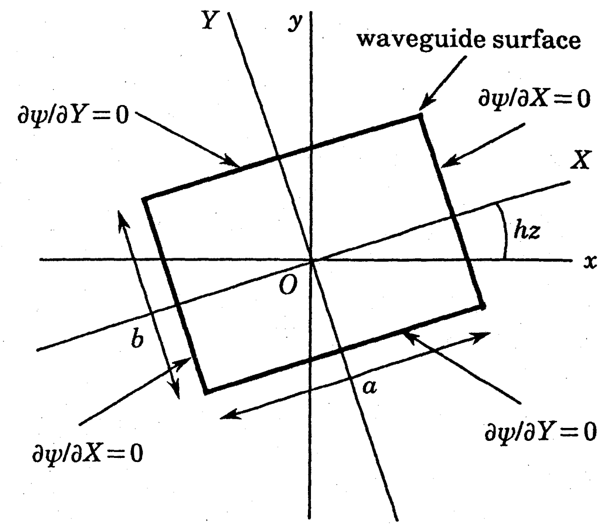

Fig. 5 Cross section of a twisted waveguide

solutions by the perturbation method [2] and FEM [12] are also shown in this table. The discrepancies between the wave numbers by BEM and those by FEM are seemed to be negligible. On the other hand, there are significant differences between the wave number by perturbation method and that by other two methods for $h=0.9$ and $b=0.8$.

A more rigorous analysis of the twisted waveguide is reported in ref. [13].

<u>MHD equilibria in a straight helical vessel ($k=\beta=0$)</u>

The MHD equilibria are described by

$$\mathbf{J}\times\mathbf{B} = \nabla p, \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J}, \quad \nabla\cdot\mathbf{B} = 0, \tag{20}$$

where $\mathbf{J}$ is the current density, $\mathbf{B}$ the magnetic field, p the plasma pressure and μ_0 the permeability of the free space. If an equilibrium configuration has a helical

Table 2. Cut-off wave numbers for a twisted waveguide ($a=1.0$)

b	Helical pitch h	Perturbation method	FEM	BEM
0.3	0.1	3.143	3.146	3.148
0.3	0.5	3.164	3.168	3.169
0.3	0.9	3.214	3.219	3.220
0.8	0.1	3.143	3.146	3.150
0.8	0.5	3.178	3.184	3.188
0.8	0.9	3.235	3.268	3.275

symmetry ($\beta = 0$), eq. (20) can be written in terms of the stream function ψ and current potential $g(\psi)$ in the following form [1, 14],

$$\nabla \cdot (K \nabla \psi) = \frac{-2K^2}{h} g(\psi) - K g(\psi) g'(\psi) - \mu_0 p'(\psi), \tag{21}$$

where $K \equiv h^2/(1 + h^2 r^2)$ and the functions ψ and $g(\psi)$ are defined by

$$B_r = -\frac{h}{r}\frac{\partial \psi}{\partial \zeta}, \quad B_\theta - hrB_z = h\frac{\partial \psi}{\partial r}, \quad hrB_\theta + B_z = hg(\psi). \tag{22}$$

The MHD equilibrium can be obtained by solving (21) for given functions $p(\psi)$ and $g(\psi)$. The principal solution ψ^* for this case is given by [1]

$$\psi^* = -\frac{2}{h^2}\left(\log r_> + \frac{1}{2}h^2 r_>^2\right)$$
$$- 4rr_0 \sum_{m=1}^{\infty} I_m'(mhr_<) K_m'(mhr_>) \cos[m(\zeta - \zeta_0)] \tag{23}$$

The BIE for eq. (21) can be readily obtained by setting the constants k and β to zero in eq. (14), which can be solved without iterations when the inhomogeneous terms in eq. (21) are independent from the function ψ, *i.e.*, $f = f(X, Y)$ (vacuum fields). Figure 6 shows the magnetic surfaces (contours of ψ) obtained by BEM [5] for vacuum fields in a conducting vessel with a circular cross section with a shift S from the origin. When the inhomogeneous terms in eq. (21) are the functions of ψ, BIE (14) must be solved using appropriate iterative techniques [15, 16].

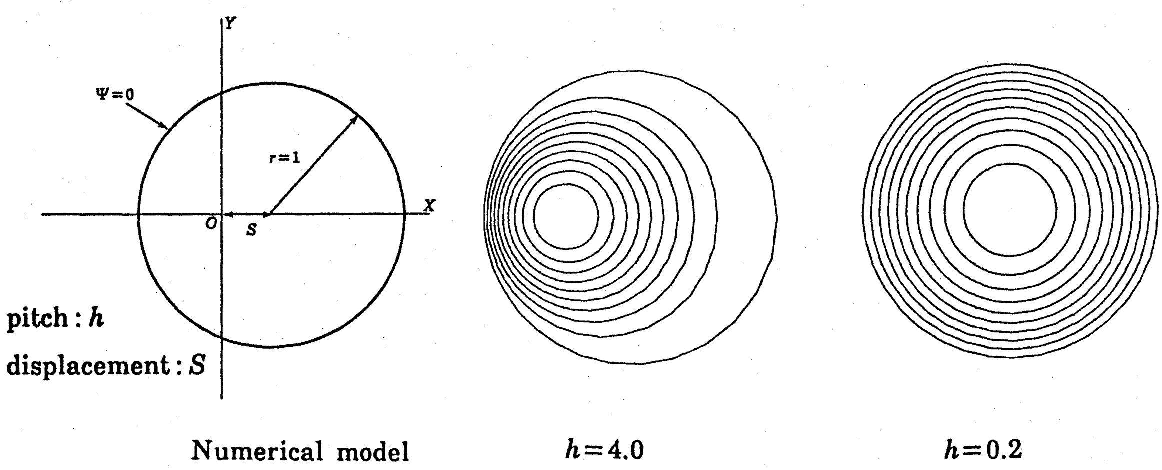

Fig. 6 MHD equilibrium configuration ($S = 0.5$)

SUMMARY

In this paper, the two dimensional integral equations have been introduced from the scalar Helmholtz-type equation in the general curvilinear coordinates and the twisted coordinates. The boundary element method for helical symmetry has been applied to the analyses of potential fields in a helical column, electromagnetic fields in a twisted waveguide and MHD equilibria in a helical vessel. We conclude that BEM is not only useful to analyze electromagnetic fields with translational and axial symmetries but also with helical symmetry.

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